An Adaptive Triangulation Refinement Scheme and Construction

Ronaldo Marinho Persiano
João Luiz Dihl Comba
Valéria Barbalho

Laboratório de Computação Gráfica
COPPE/UFRJ
C.P. 68511 Rio de Janeiro
21945 RJ, Brasil
marinho@lcg.ufrj.br

Abstract. A general scheme for adaptive triangulation refinements is described where the simplices of 3D triangulation are bisected as part of the refinement. The refinements are shown to be minimal in the number of elements. The scheme is dependent only on a refinement criterion that is application dependent. Examples of application to piecewise-linear approximation of implicit surfaces and finite-elements are briefly discussed. A sweeping version of the scheme is described.

The recent years have shown an increasing interest in simplicial subdivision schemes to solve geometrical problems, system of algebraic equations and differential problems in 3D space and even in n-space. Simplicial subdivision has been applied to find the fixed points of transforms, to define meshes for finite element methods (RIVARA[1991], NAMBIAR[1993]), to track implicitly defined curves and surfaces in Euclidean spaces (ALLGOWER[1987], MIRANDA[89], SALIM[1991], CASTELO[1990]) and to determine a boundary representation of CSG solids (PERSIANO[1991]). However, many simplicial subdivision schemes are either regular, that is, subdivide the space in equivalent cells, or irregular but a triangulation structure is not assured. In this paper we will discuss a class of nonregular triangulations derived from the bisections of CFK triangulations. A general scheme for the construction of such triangulations will be presented and shown to have optimality properties. Finally, a sweeping version of this scheme will be presented.

1. Regular CFK Triangulations

Three-dimensional simplices are non degenerated tetrahedra. They are the more simple volumetric form in three-dimensional space comparable only to cubes in simplicity and utility. Cubes have been used in grid arrays and in recursive subdivisions, like in octrees, mainly due to simple orientation in respect to Cartesian coordinate systems. We will consider simplicial subdivisions of cubic portions of the space in the sequel.

A cube may be subdivided by three planar cuts in 6 similar tetrahedra sharing its vertices and one of the cube diagonals (fig. 1). The cube so subdivided will present each of its faces subdivided in two triangles by their diagonals. From a regular subdivision of the space in cubes, regular simplicial subdivision for the whole space may be defined from the subdivision of the cubes. If the subdivision of a cube in 6 simplices is translated to every other cube of the cubic mesh we obtain a *K1 triangulation* of the space. If, instead, the basic subdivision of a cube is reflected by its faces to its neighbors, a *J1 triangulation* of the space is formed. These two schemes assure a *conform simplicial subdivision*, that is, the intersection of any two simplices either is empty or is a vertex, or a full edge or a full face of both simplices. Conform simplicial subdivisions are generically called *triangulations*. The K1 and J1 triangulations have been called *Coxeter-Freudenthal-Kuhn (CFK)* (MIRANDA[1989]) and have been used in several applications (ALLGOWER[1987], CASTELO[1990] , SALIM[1991], PERSIANO[1991]). See fig. 2 for examples of K1 and J1 triangulations in 2D .

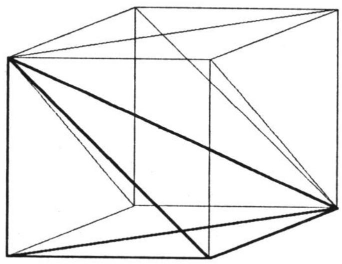

Figure 1: The basic subdivision of a cube in 6 similar tetrahedra

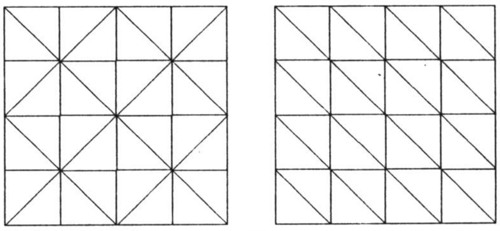

Figure 2: The J1 and K1 CFK triangulations in 2D.

2. Simplex Bisections and Variations of CFK Triangulations

Two variations of the above CFK triangulations may be built by simple refinement of its simplices by a *halving bisection procedure*. A halving bisection of a simplex is attained by cutting it by the plane passing by the midpoint of its longest edge containing its opposed edge.

Each simplex of a J1 or K1 CFK triangulation has the diagonal of the cube as its longest edge and some of the edges of the cube as its smallest edges. The extensive application of the halving bisection procedure to all simplices of a J1 or K1 triangulation will lead to a new triangulation that we will call a *S1 triangulation*. The new vertices and faces inserted in the original triangulation will all be contained in the interior of the cubes (fig. 3a).

The longest edge of a S1 triangulation simplex is a diagonal of a face of its cube and their smallest edges are halves of cube diagonals. Another variation of CFK triangulation results by applying the same halving bisection procedure to each simplex of a S1 triangulation. The resulting triangulation will be called a *R1 triangulation* (fig. 3b). If another general bisection is applied to a R1 triangulation then a finer J1 triangulation is again achieved (independently whether we started from a J1 or K1 triangulation) (see fig. 3c).

Note that the clusters of simplices of a J1 or K1 triangulation in 3D having the same longest edge form cubes. For S1 triangulations, the clusters of simplices having the same longest edge form octahedra, each one contained in two original cubes. For R1 triangulation, these clusters are also octahedra but contained in four cubes (fig. 3). In any case, the halving bisections of the simplices in a cluster will insert edges and faces restricted to the interior of that cluster.

All simplices of S1 (or R1) triangulations have the same shape being either a translation or a reflection (followed by a translation) of any other one. However, although J1 and K1 triangulation simplices possess the same shape, S1 and R1 simplices have different shapes. Therefore, the successive application of the general bisection procedure produces only three shapes of simplices denominated here *types A (for J1 and K1), types B (for S1) and types C (for R1)*. Note that the bisection of a simplex of type A generates two simplices of type B, whose bisection generates two simplices of type C, whose bisection regenerates two type-A simplices again.

The simplices of these three types have reasonable shape in the sense they are relatively "fat" simplices. We may measure how "fat" a tetrahedron is by evaluating its minimum internal solid angle. Fatter tetrahedra will have greater measure. Regular tetrahedra are the fattest by that measure and degenerated tetrahedra have zero measure.

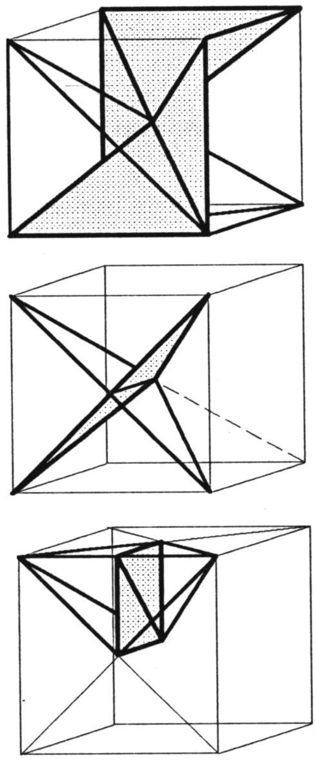

Fig. 3: Successive bisections of simplices of a CFK triangulation

The table below presents the values in degrees of the internal angles of the three types of tetrahedra A, B and C. Recall that the internal angle of a regular tetrahedron is 45 degrees. Tetrahedra of types A, B and C all have the same minimum internal solid angle that is 1/3 of the measure of regular tetrahedra.

Type A	Type B	Type C
15	15	15
45	60	90
45	30	30
15	15	15

Figure 2 shows the corresponding J1 and K1 CFK triangulations in 2D space. Note that all triangles have the same format. A S1 triangulation in 2D has similar triangles but rotated 45 degrees in a distinct arrangement. The general bisection of S1 triangulations produces a J1 triangulation. Therefore, in 2D space there are only two types of triangles generated by the general bisection.

Hierarchical Simplicial Subdivision

A simplex may be subdivided in many ways, so leading to several recursive subdivisions. The halving bisection of a simplex introduced in the previous section leads to a hierarchical binary subdivision scheme (fig. 4) that may be adapted to local requirements in contrast to the widespread uniform refinement of S1 and R1 triangulations.

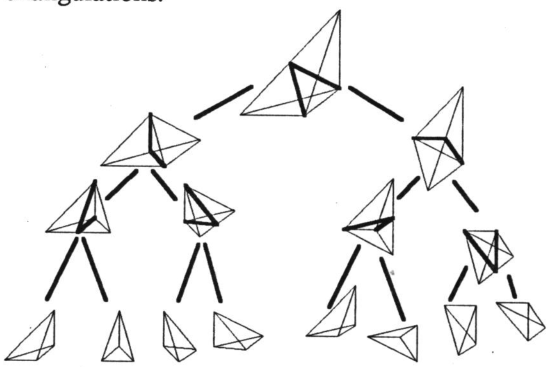

Fig. 4: Binary subdivision tree of a CFK simplex.

Adaptive simplicial subdivisions based on CFK triangulations have been used to reduce the complexity of the subdivisions. SALIM[1991] reports an adaptive simplicial subdivision where the halving bisections of simplices are locally guided by a root search criterion. However, Salim's simplicial subdivision is by no means a triangulation that is unacceptable to other applications like finite-element mesh generation and piecewise-linear approximation of three-variate real-valued functions.

3. Adaptive Triangulation Refinement

Subdivision Refinement

We call the *domain of a subdivision* the union of all its cells. Given any two subdivisions T1 and T2, we say that T2 is a *subdivision refinement* of T1 if the domain of T2 equals the domain of T1 and each cell of T2 is fully contained in some cell of T1. A *triangulation refinement* is a subdivision refinement of a triangulation that happens to be a triangulation. The refinement of a triangulation is the result of cutting some of its simplices in two or more simplices. In a *binary refinement* of a triangulation the refinement of a simplex is performed by successive binary cuts. S1 and R1 triangulations are examples of binary refinements of J1 or K1 triangulations. Note that hierarchical subdivisions are examples of subdivision refinements of triangulations but are not triangulation refinements.

It is possible to associate a binary subdivision tree to any binary refinement of a triangulation: the branches of the tree correspond to the binary cuts and the leaves of the tree correspond to the simplices of the refined triangulation. For any simplex of a binary refinement we may associate the depth of the corresponding leaf of its binary subdivision tree.

Adaptive Refinements of CFK Triangulations

CFK triangulations have many interesting combinatorial properties and have been the basis for several triangulation techniques in n-dimensional spaces. However, if regular triangulation is used the depth of refinement (size of simplices) should be adjusted by the critical spot of the problem space and a huge triangulation may result. Ideally, a (not necessarily regular) triangulation could be tailored to the problem in such a way that each simplex is just small enough to ensure that a given precision criterion be satisfied. For example, if the triangulation is intended to ground a piecewise-linear approximation of a three-variate function, some local linearization criteria (possibly based on second order derivatives) might guide the choice of the maximum size of simplex that is locally allowed.

Binary refinements of CFK triangulations based on halving bisections may profit from the reasonable shape of the simplices that is essential for applications like the one above. Ideally we may ask for the better (less complex) binary refinement of an initial CFK triangulation that suits the problem requirements. This certainly encompasses a combinatorial optimization problem that may be too complex in the general case. Alternatively, a sequence of optimal small steps of refinement may lead to a good sub-optimal solution.

A procedure to refine general triangulations by halving bisections was proposed by RIVARA[1991] for application to the generation of finite-element meshes. Her procedure determines a triangulation refinement resultant from halving simplices that accommodates the halving bisection of a given simplex. Rivara's refinement is recursive and may be rewritten the following way:

Basic Refinement Procedure of Simplex S of triangulation T

Step 1: let E be the longest edge of S
while E is not the longest edge of some simplex S' of T containing E
do a basic refinement of S';
Step 2: do a halving bisection of all simplices containing E.

The basic refinement procedure has very useful properties. It is a finite algorithm because at each recursive call the longest edge of the simplex S is greater than the longest edge of the previous call. The goal of the procedure is achieved by recursively determining what to do and bisecting the simplices just in the backtracking. The only step that produces a change in the triangulation is the step 2 that ensures the triangulation structure is preserved; therefore if we perform this step in a single execution we will always have a (conforming) triangulation (and no inconsistent data structure may be generated). Observe that for general triangulations the number of iterations of the while loop of the step 1 is dependent on the geometry of the simplices that are incident to E and may be performed many times if the triangulation contains excessively thin simplices.

We will call *basic (binary) refinement of (S,T)* the triangulation generated by applying the basic refinement procedure to (S,T). An *(CFK-based) adaptive triangulation* is any triangulation generated by successive applications of the basic refinement procedure to simplices of a (K1 or J1) CFK triangulation. Certainly, due to the properties of the simplices of CFK triangulations, the simplices of a three-dimensional adaptive refinement all have one of the three shapes of types A, B or C described above and therefore the measure of the initial simplices is preserved (which is not true in general triangulations).

We claim that the basic refinement of (S,T) solves the problem of finding the best (minimal) binary refinement of the triangulation T that bisects the simplex S. Therefore, any other binary triangulation refinement that bisects S will be a refinement of the basic refinement of (S,T). In this sense, the basic refinement is the minimal step in any scheme of binary triangulation refinement by halving bisections.

Furthermore, if we denote by B(S,T) the basic refinement of (S,T), then we may verify that for any simplices S1 and S2 of T the two triangulations B(S2,B(S1,T)) and B(S1,B(S2,T)) are equal. Therefore, the triangulation refinement by halving bisections is *order invariant*.

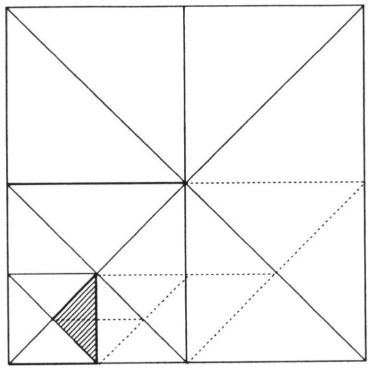

Fig. 5: An adaptive triangulation in 2D.

Figure 5 shows an example of a 2D adaptive refinement where the dotted segments represent the edges the basic refinement procedure would insert in the triangulation to bisect the hatched triangle. It is easy to see that in 2D adaptive refinements the basic refinement will ever follow a unique path of triangles to bisect. In the 3D case, however, many "branches" may be followed by the procedure whose "ends" are clusters of simplices of the same type meeting at their longest edge. Besides, in the 2D case, the length of the longest edge at each recursive call is increased by a factor of square root of 2, and a maximum of n simplices is introduced in the triangulation by each application of the basic refinement, where n is the subdivision depth.

Adaptive Refinement Construction

Triangulations are used as a support to many applications (see section 4). Often, the application demands the triangulation be refined enough to guarantee that some problem requirements are fulfilled. Adaptive triangulations are suitable for problems whose refinement requirements vary throughout the problem space domain.

Now suppose that the requirements of the application problem is *spatially decomposable*, that is they may be expressed by some criterion that evaluates whether a simplex of a triangulation is acceptable or not. The whole triangulation will satisfy the problem requirements if all its simplices are accepted by the criterion. Given any triangulation T for the problem then we may ask for a (minimal) refinement of T that is accepted by the refinement criterion.

Spatially decomposable requirements are often expressed by *invariant-to-inclusion criteria* that is if a simplex is approved by the criterion then any simplex contained in it will also be approved. Invariance to inclusion is a property of the problem that may expressed by "the finer the better".

A simple scheme may be proposed to build refined triangulations for solving problems whose requirements are spatially decomposable and expressed by a criterion invariant to inclusions:

Step 1: Start with a CFK triangulation T covering the problem domain;
Step 2: while T has some simplex S not accepted by the criterion do a basic refinement of (S,T).

It is easy to see that the simple procedure above obtain an acceptable solution (and stops) if there is some refinement of the initial triangulation T that fulfills the problem requirements. We claim in addition that the final refined triangulation is minimal, that is it is the less complex refinement of T acceptable by the criterion. This optimality property of the refined solution is a consequence of the minimality of basic refinement procedure and the invariance to inclusions of the criterion. Note that the *optimal refinement scheme* above does not require any specific order for applying the basic refinement: once any unacceptable simplex is detected the basic refinement is applied to it. Specific ordering of the criterion tests and of basic refinement may lead to algorithms with distinct features. In section 5, a specific ordering will be used in a sweeping version of the optimal refinement scheme.

4. Adaptive Triangulation Applications

Adaptive triangulations are appropriated for problems whose solution is spatially nonuniform and where some kind of local linearization leads to a reasonable approximation. Typically, the schemes based on adaptive triangulation break the space in small portions and build for each portion an approximate (possibly linear) solution to the problem in that portion of space.

Implicitly Defined Surface Evaluation

Consider a surface defined by an equation $f(x,y,z)=0$ where f is a continuous three-variate real function. The f values at the vertices of a given triangulation define a unique affine function in each simplex that interpolates f at the vertices of the simplex. The piecewise-linear function f_Δ so induced in the whole triangulation is continuous for the interpolants corresponding to two simplices sharing the same face have the same values in that face. Therefore if f is smooth enough and the triangulation is fine enough then f_Δ will be a good approximation of f on the domain of the triangulation (ALLGOWER[1985]).

If a precision is prescribed for the approximation, a criterion may be set to evaluate whether that precision is attained in a given simplex or not. Such criterion may, for example, consider the maximum value of the second order derivatives to estimate the error the linearization will produce and compare it with the precision level. This kind of criterion is certainly invariant to inclusions and therefore meets the requirements of applicability and optimality of the optimal refinement scheme of the previous section. The final triangulation generated by this method will be minimal and formed by well-shaped tetrahedra.

Boundary Evaluation of CSG Solids

The boundary of regular CSG solids may be described as the roots of a real valued three-variate function if the CSG primitives are half-spaces. Therefore, the general scheme discussed above may be tailored to evaluate the boundary of CSG solids.

A scheme similar to PERSIANO[92] may be designed, guided, however, by a refinement criterion that takes into account the "curvature" of the boundary to stop the level refinement when the linearization is good enough. As a consequence of using a curvature-dependent criterion, the polyhedral approximation of the solid boundary will contain smaller polygons only on portions of the surface of high curvature or near the solid edges. An adaptive boundary evaluation of CSG solids based on these ideas is being studied (APOLINÁRIO[1993]).

Finite Element Mesh Construction

Adaptive methods for finite element analysis are techniques that interleave the generation of the finite element mesh and the finite element analysis (RIVARA[1990], NAMBIAR[1993], DEVLOO[1991], LÖHNER[1992]). Typically, starting from an initial mesh, a finite element analysis is carried on producing an estimate of the local error due to the mesh geometry. Based on this error estimates a new mesh is generated by refining the elements where the errors are unacceptable, and a new analysis is done. The method iterates these two phases until an adequate mesh for the problem is obtained.

The basic refinement introduced by Rivara[1990] was originally proposed as part of an adaptive finite element technique. When a complex geometric model is under analysis, the triangulation refinement should take into account both the geometric approximation of the model and the adequacy of the triangulation for the phenomena being studied. Barbalho[93] is currently developing a finite element mesh generator for adaptive finite element analysis of three-dimensional models described by CSG expressions. Adaptive refinement of CFK triangulations seems to meet all requirements for these applications.

5. Adaptive Triangulation Refinement by Sweeping

The optimal refinement scheme of section 3 operates basically scanning the current triangulation refinement looking for a simplex not accepted by the criterion and applying to it the basic refinement. Although the order of this search and basic refinements are not relevant to the final result, an organization of the simplices to be scanned for testing may avoid unnecessary criterion tests and reduce lists of candidate simplices.

The algorithm described in this section implements the optimal refinement scheme for 2D triangulations by progressing the test and basic refinement in a single sweeping of the triangulation. The sweeping frontier will advance from left to right in the plane and at any time will delimit the region (to its left) of triangles already tested and accepted by the criterion. Triangles to be tested will basically remain to the right of the frontier. Hence, when the frontier reaches the right border of the triangulation domain, the refinement is concluded and all simplices were approved by the criterion.

The strategy of the algorithm was inspired in topological sweeping techniques like the one proposed by EDELSBRUNNER[1986] where a topological line, not necessarily straight, is used to sweep a collection of figures in the plane. The sweeping frontier in our case, however, will not be necessarily continuous.

The Sweeping Algorithm

Suppose the initial CFK triangulation domain is a rectangle in the plane. The basic idea of the algorithm is to sweep the triangulation from left to right, partitioning triangles (by basic refinements) when the evaluation criterion requires. The sweeping frontier (also called sweeping list) will be a collection of triangle edges such as their projection on the right border of the domain covers that border.

In addition to the triangulation itself, the sweeping list will be the only data structure the algorithm will keep to perform its task. While the sweeping list is not empty, the algorithm iteratively removes an edge E from the list, determines the triangle T to the right of E, and if T has not been accepted yet, tests it against the criterion, and depending on the result performs one of the following steps:

> *Advance step*: when the triangle T is accepted by the criterion the sweeping frontier must advance; the rightmost (nonhorizontal) edges of the triangle are inserted in the sweeping list.
>
> *Refinement step*: otherwise, a basic refinement is applied to the triangle T; the edge E (or its refinement) is reinserted in the sweeping list; and if, during the propagation of the basic refinement, any edge of the sweeping list suffers a bisection, then it is removed from the list and its bisected parts are inserted instead.

Note that during the process as long as the triangles are accepted in the advance step, they are marked in order to avoid that the triangle be processed again in a further iteration. From the above description, it is clear that we should keep for each edge in the sweeping list the triangle to its right and for each triangle we should be able to determine its rightmost edges. In order to support these demands, the orientation of the triangles is the main key.

In a 2D CFK triangulation refinement, all triangles are right triangles and may assume 8 orientations classified according to the direction of its right angle: N, S, E, W, NE, NW, SE, and SW. Keeping a record of the orientation of the triangles and a convenient enumeration of its edges enable us to determine the rightmost edges of a triangle by a table-driven method. The orientations and edge enumerations of two triangles generated by a bisection may also be tabulated as a function of the orientation of their father.

Figure 6 shows the refinement of a CFK triangulation after some iterations of the sweeping method. In this example, the criterion approves a triangle if either it does not contains any root of the equation $x^7 - y^5 + x^2y^3 - xy^2 = 0$ or it is smaller than a prescribed size. The final refined triangulation illustrated in figure 7 has 4664 triangles.

3D Sweeping Algorithm

The generalization of the sweeping refinement algorithm to the 3D case is almost straightforward. The sweeping frontier is now a surface that travels from the left border of the domain to the right border and the sweeping list is composed of tetrahedron faces. No modification is needed in the previous formulation of the algorithm in two steps.

Certainly, the number of tetrahedron positions, shapes and orientations is greater than the 8 instances of the 2D case but there is no additional complexity. The simple combinatorics of the CFK triangulation refinements allows that tables may be built to drive the steps of the method .

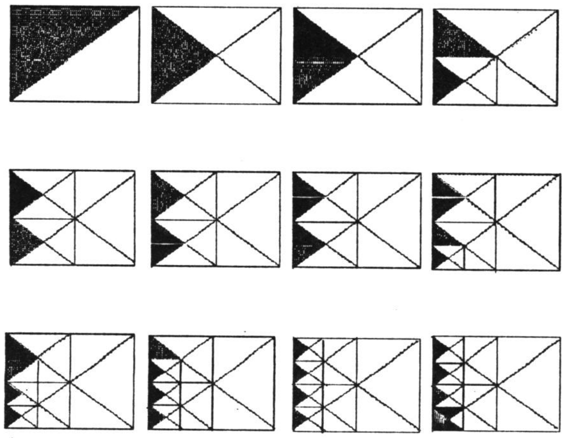

Fig. 6: Initial steps of the sweeping algorithm

Conclusions

We presented here a general scheme for simplicial subdivisions suitable for adaptive refinements. Thanks to the one-step optimality of the basic refinement procedure and its order invariance we are able to assure that any refinement generated by a sequence of applications of the basic refinement will be optimal. On the other hand, the good shape of the bisections of CFK simplices enables us to produce good triangulations from refinements of a given CFK triangulation.

Based on those properties we discussed a general refinement scheme based on a simplex-based refinement criterion. That scheme allows that triangulation refinements be built by the simple application of the basic refinement procedure to every remaining simplex not approved by the refinement criterion. If the criterion satisfies a reasonable assumption of invariance to inclusion, it is possible to guarantee the optimality of the final refinement.

Applications of that general scheme to implicitly-defined surface evaluation, to boundary evaluation of CSG solids and finite element mesh construction illustrated its potential. A sweeping version of the general scheme were discussed.

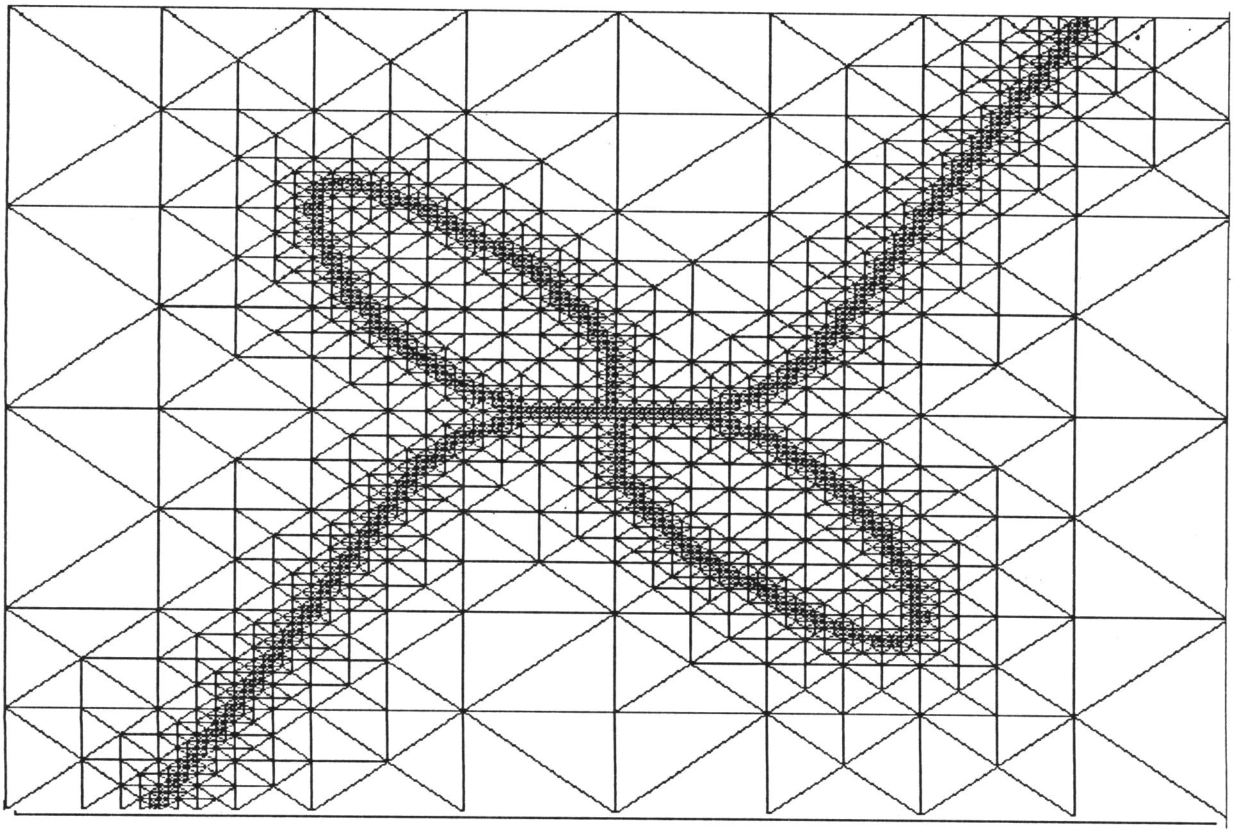

Fig. 7: Final refinement generated by the sweeping algorithm.

Acknowledgments

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References

ALLGOWER, E. and SCHMIDT, P.: An Algorithm for Piecewise-Linear Approximation for an Implicit Defined Manifold. *SIAM Journal on Numerical Analysis*, vol. 22, n. 2, 1985.

ALLGOWER, E. and GNUTZMANN, S: An Algorithm for Piecewise-Linear Approximation for an Implicit Defined Two-dimensional Manifolds. *SIAM Journal on Numerical Analysis*, vol. 24, n.2, 1987.

APOLINÁRIO, A: Boundary Evaluation of CSG Solids by Adaptive Refinement. M.Sc. Thesis, *COPPE/UFRJ*, 1993 (in preparation)

BARBALHO, V.: Adaptive Finite Element Mesh Generation of CSG Solids. M.Sc. Thesis, *COPPE/UFRJ*, 1993 (in preparation).

CASTELO, A., FREITAS, S. and TAVARES, G.: Approximation to Manifold and its Application to Implicit ODES. *Lectures in Applied Mathematics*, n. 26, pp. 29, AMS, 1990.

DEVLOO, P.: A Three-dimensional Adaptive Finite Element Strategy. *Computers & Structures*, vol. 38, n. 2, pp. 121-130, 1991.

EDELSBRUNNER, H. and GUIBAS, L.: Topologically Sweeping an Arrangement. DEC Systems Research Center, Technical Report 9, April, 1986.

LÖHNER, R. and BAUM, J.: Adaptive h-refinement on 3D Unstructured Grids for Transient Problems. *International Journal for Numerical Methods in Fluids*, vol. 14, pp. 1407-1419, 1992.

MIRANDA, J. and TAVARES, G.: Métodos Simpliciais em Computação Gráfica. *17o Colóquio Brasileiro de Matemática*, 1989.

NAMBIAR, R. et al.: An Algorithm for Adaptive Refinement of Triangular Element Meshes. *International Journal for Numerical Methods in Engineering*, vol. 36, pp. 499-509, 1993.

SALIM, M., BUENO, L. P. and PERSIANO, R.: Aproximação Linear por Partes de Variedades Implicitamente Definidas Usando Poda no Espaço. *Anais do IV Simpósio Brasileiro de Computação Gráfica e Processamento de Imagens (SIBGRAPI)*, p. 27, 1991 (in Portuguese).

PERSIANO, R., SALIM, M. and BUENO, L. P.: Boundary Evaluation of CSG Solids by Simplicial Subdivision. *COMPUGRAPHICS'91*, Sesimbra, Portugal, Oct, 1991.

RIVARA, M. and LEVIN, C.: A 3D Refinement Algorithm Suitable for Adaptive and Multigrid Techniques. *Communications in Applied Numerical Methods*, vol. 8, pp. 281-290, 1992.

RIVARA, M.: Local Modification of Meshes for Adaptive and/or Multigrid Finite-element Methods. *Journal of Computational and Applied Mathematics*, 36, pp. 79-89, 1991.